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Oscillation of a bubble in a liquid confined in an elastic solid

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A simple theoretical model is described for the oscillation of a gas bubble in a liquid in a cavity confined by an elastic solid. The phenomenon occurs in nature and technology but has only recently received attention. The compressibility effects in the continuity equation are shown to be negligible, using dimensional analysis. However, the volume change of the confined liquid has to be considered since the associated pressure variation is large. The variation of the cavity volume is assumed to be proportional to the change of the liquid pressure at the confinement wall. The Rayleigh-Plesset-like equation describing the dynamics of a confined bubble is derived, considering the viscous and surface tension effects. Using the linear stability analysis, we show that the bubble undergoes stable damping oscillation when it is subject to small disturbances. The natural frequency of oscillation is obtained analytically. The theory agrees well with recent experiments. The analyses show that the natural frequency of oscillation for a bubble in an elastic confinement is larger, in order of magnitude, than that in an unbounded liquid. The amplitude and period of oscillation of a transient bubble decrease significantly owing to the presence of a confinement, reaching a steady state in a much longer period and at a larger equilibrium radius. When subject to an acoustic wave, a bubble in a confinement oscillates at smaller amplitude. The effects of the confinement increase with the bulk modulus of the confinement and decrease rapidly with the cavity size but are still significant for a large cavity whose size is an order of magnitude larger than the bubble. *Published by AIP Publishing.*
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I. INTRODUCTION

Bubble dynamics have been investigated extensively, for a bubble in an infinite fluid domain,[1–5] near a rigid surface,[6–9] or in a tube.[10] Studies were also carried out for the interaction of a bubble and a free surface,[11–13] an elastic surface,[14] or another bubble.[15] Bubble dynamics in non-Newtonian fluids were analyzed for spherical bubbles[16] and non-spherical bubbles.[17] Reviews on bubble dynamics can be found in Refs. 18–23.

However, little attention has been given to bubbles oscillating in a liquid inside a cavity in a solid. Here, a cavity denotes a confined space within a solid fully confined in all directions. Confined bubbles occur in natural and engineering applications. Plants contain fluid-filled vessels (xylem) to transport water from the roots to the leaves. Vulnerability to drought-induced cavitation is a key trait of plant xylem. Growth of cavitation bubbles in plant xylem is believed to be a cause of mortality during drought.[24–26] Confined bubble dynamics in plants are associated with ejection of spores in some species of ferns.[27] Acoustic emission of pressure waves is associated with cavitation in plant xylem and their applications in plant sciences are rapidly increasing, especially to investigate drought-induced plant stress.[25] Liquids trapped in rocks may contain bubbles and their quasi-static behaviour is used by geologists to extract past thermodynamic conditions.[28,29] Other applications of such a non-invasive technique include cavitation monitoring in porous media and quartz inclusions or soil tensiometers.[30]

Recently, Vincent *et al.*[30–32] carried out a series of interesting experiments for the dynamics of a cavitation bubble in a liquid fully entrapped in a transparent elastic solid, using light scattering, laser strobe photography, and high speed camera recordings. Their experiments showed unexpectedly fast bubble oscillations in volume, depending on the confinement size and elasticity. They also observed rich non-spherical dynamics, with ejection away from the walls and bubble fragmentation. In particular, they analyzed the natural frequency of oscillation of a confined bubble using Minnaert's theory, by considering the compressibility of the liquid and elasticity of the confining solid. They calculated the kinetic energy of the liquid flow induced by a bubble in a confinement using the formula for a bubble in an unbounded incompressible liquid, which leads to an error at the order of the ratio of the bubble radius to the cavity radius, as shown in Sec. II.

While a bubble oscillates in a confinement, the liquid pressure changes at much larger amplitude than in an infinite fluid. Using the dimensionless analysis, the compressibility effects of water are shown to be negligible in the dimensionless continuity equation. When the variation amplitude of the liquid pressure is at 20 MPa, the compressibility effects in the continuity equation are at the order of 0.01. But this high pressure variation changes the volume of the confined liquid, which plays a role in the dynamics of a confined bubble.[30] A fully confined bubble can grow by compressing the liquid around it and expanding/pushing away the confining solid.

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With the above considerations, we describe a simple physical and mathematical model for the bubble dynamics in a spherical elastic confinement, considering the interaction among the expansion/collapse of the bubble gas, the liquid flow induced, and the deformation of the elastic confinement. We assume that a gas bubble is under adiabatic or isothermal condition, the liquid flow is described by the incompressible continuity equation, and the volume variations of the liquid and the cavity are both linear to the pressure variation of the liquid at the interface between the liquid and solid. The Rayleigh-Plesset-like equation is described for a bubble in a confinement considering the viscous and surface tension effects. We perform the linear stability analysis for a confined bubble and obtain its natural frequency of oscillation. The computed frequency of a confined bubble and the radius history of a transient confined bubble agree well with the recent experimental results.[30] The natural frequency, transient oscillation starting from a non-equilibrium state, and oscillation of a bubble subject to an acoustic wave are analyzed, in terms of the confinement size and bulk elasticity.

II. RAYLEIGH–PLESSET-LIKE EQUATION FOR A CONFINED BUBBLE

Consider a bubble in a liquid in a cavity in an elastic solid. Here cavity denotes a confined space within a solid as shown in Fig. 1. The system consists of the internal gas/vapor of the bubble, the liquid in the cavity, and the elastic solid surrounding the cavity. To facilitate the mathematical analysis, the bubble and cavity are assumed to be spherical and concentric. We denote the instantaneous radii of the bubble and cavity as $R_b(t)$ and $R_c(t)$, respectively, where t denotes the time.

In the spherical coordinates with the origin at the centre of symmetry, the continuity equation of the liquid flow reads

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r^2}\frac{\partial\left(r^2 u\right)}{\partial r} = -\frac{1}{\rho}\frac{d\rho}{dt}, \tag{2.1}$$

where ρ is density, $\boldsymbol{u}$ and u are the velocity of the liquid flow and its radial component, respectively, r is the radial distance measured from the bubble centre to a field point, and d/dt denotes the material derivative. Since $\frac{d\rho}{dt} = \frac{d\rho}{dp_l}\frac{dp_l}{dt}$ and $\frac{dp_l}{d\rho} = c^2$, where p_l is the pressure and c is the sound of speed in liquid, we have

$$\frac{1}{r^2}\frac{\partial\left(r^2 u\right)}{\partial r} = -\frac{1}{c^2\rho}\frac{dp_l}{dt}. \tag{2.2}$$

We choose the initial bubble radius R_{b0} as the reference length, the density ρ_0 of the undisturbed liquid as the reference density, and $\Delta p = p_{l0} - p_{sat}$ as the reference pressure, where p_{l0} is the initial liquid pressure and p_{sat} is the saturated vapor pressure. Equation (2.2) can be written in the dimensionless form as follows, with the dimensionless quantities indicated by an asterisk subscript:

$$\frac{1}{r_*^2}\frac{\partial\left(r_*^2 u_*\right)}{\partial r_*} = -\frac{\Delta p}{\rho_0 c^2}\frac{1}{\rho_*}\frac{dp_*}{dt_*}. \tag{2.3}$$

If the reference pressure $\Delta p \le 2,\ 20$ MPa, we have $\Delta p/\left(\rho_0 c^2\right) \le 0.001,\ 0.01$, respectively, since $c \approx 1500$ m s^{-1} and $\rho_0 \approx 1000$ kg m^{-3} for water. If reference pressure $\Delta p \le 20$ MPa, the right hand term in (2.3) is negligible, i.e., $\partial\left(r^2 u\right)/\partial r = 0$, so that

$$u(r,t) = \frac{R_b^2 \dot{R}_b}{r^2} = \frac{a(t)}{r^2}, \quad a(t) = R_b^2 \dot{R}_b, \tag{2.4}$$

where the over dot denotes the derivative in time t.

Using (2.4), the kinetic energy of the liquid in the cavity can be calculated as follows:

$$\begin{aligned} E_k &= \frac{1}{2}\rho\int_{R_b}^{R_c} 4\pi r^2 u^2 dr = 2\pi\rho R_b^3 \dot{R}_b^2\left(1 - \frac{R_b}{R_c}\right) \\ &= E_{k\infty}\left(1 - \frac{R_b}{R_c}\right), \end{aligned} \tag{2.5}$$

where $E_{k\infty} = 2\pi\rho R_b^3 \dot{R}_b^2$ is the kinetic energy of a bubble in an infinite incompressible fluid.[20,30] Using $E_{k\infty}$ to approximate E_k results in an error of the order of R_b/R_c.

It is assumed that the volume of the bubble is small compared to the volume of the cavity, i.e., $V_b/V_c = (R_b/R_c)^3 \ll 1$.[30] As such, the volume variation of the cavity is assumed to be linear to the pressure variation at the confinement wall,

$$V_c - V_{c0} = \frac{V_{c0}}{K_c}\left(p_{lc} - p_{l0}\right), \tag{2.6}$$

where V_{c0} and V_c are the initial and transient volumes of the cavity, respectively, p_{lc} is the transient liquid pressure at the interface between the liquid and solid, $p_{lc} = p_l(R_c, t)$, and K_c is the bulk modulus of the elastic confinement.

It is also assumed that the change of the liquid volume is proportional to the pressure variation at the confinement wall,[30]

$$V_l - V_{l0} = -\frac{V_{l0}}{K_l}\left(p_{lc} - p_{l0}\right), \tag{2.7}$$

where V_{l0} and V_l are the initial and transient volumes of the liquid, respectively, and K_l is the bulk modulus of water.

The volume of the cavity equals to the sum of the volumes of the bubble and liquid,

$$V_c = V_b + V_l, \quad V_{c0} = V_{b0} + V_{l0}, \tag{2.8}$$

where V_{b0} and V_b are the initial and transient bubble volumes, respectively. Using (2.6)–(2.8) yields

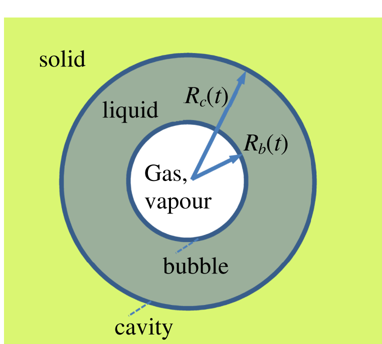

FIG. 1. Illustration of a gas/vapor bubble in liquid in a cavity confined by an elastic solid. The bubble and cavity are assumed to be spherical and concentric with their radii denoted as $R_b(t)$ and $R_c(t)$, respectively.

$$V_b - V_{b0} = \frac{V_{c0}}{K_c}(p_{lc} - p_{l0}) + \frac{V_{l0}}{K_l}(p_{lc} - p_{l0}). \tag{2.9}$$

If $V_{b0} = 0$ or $V_{c0} >> V_{b0}$, we have

$$\frac{V_b - V_{b0}}{V_{c0}} = \left(\frac{1}{K_c} + \frac{1}{K_l}\right)(p_{lc} - p_{l0}), \tag{2.10}$$

i.e.,

$$p_{lc} - p_{l0} = K\frac{V_b - V_{b0}}{V_{c0}} = K\frac{R_b^3 - R_{b0}^3}{R_{c0}^3}, \quad K = \frac{K_c K_l}{K_c + K_l}, \tag{2.11}$$

where R_{c0} is the initial cavity radius, $R_{c0} = R_c(0)$. K is termed as the effective bulk modulus for the system, which increases with the bulk modulus of liquid K_l and the bulk modulus of the confinement K_c.

Substituting (2.11) into (2.6) yields

$$V_c - V_{c0} = \frac{K}{K_c}(V_b - V_{b0}) \quad \text{or} \quad R_c^3 - R_{c0}^3 = \frac{K}{K_c}\left(R_b^3 - R_{b0}^3\right). \tag{2.12}$$

The Navier-Stokes equation for a compressible flow reads[33]

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p_l + \mu\nabla^2\boldsymbol{u} + \frac{\mu}{3}\nabla(\nabla\cdot\boldsymbol{u}), \tag{2.13}$$

where μ is the viscosity of the liquid. Its r-component is

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right) = -\frac{\partial p_l}{\partial r} + \mu\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial r^2 u}{\partial r}\right) + \frac{\mu}{3}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial r^2 u}{\partial r}\right). \tag{2.14}$$

Substituting (2.4) into (2.14) yields

$$\frac{\dot{a}}{r^2} - 2\frac{a^2}{r^5} = -\frac{1}{\rho}\frac{\partial p_l}{\partial r}. \tag{2.15}$$

Integrating the above equation to r from $R_b(t)$ to r and $R_c(t)$, respectively,

$$\dot{a}\left(\frac{1}{r} - \frac{1}{R_b}\right) - \frac{a^2}{2}\left(\frac{1}{r^4} - \frac{1}{R_b^4}\right) = \frac{p_l(r,t) - p_{lb}}{\rho}, \tag{2.16a}$$

$$\dot{a}\left(\frac{1}{R_c} - \frac{1}{R_b}\right) - \frac{a^2}{2}\left(\frac{1}{R_c^4} - \frac{1}{R_b^4}\right) = \frac{p_{lc} - p_{lb}}{\rho}, \tag{2.16b}$$

where p_{lb} is the transient pressure of the liquid at the bubble surface, $p_{lb} = p_l(R_b, t)$. Equation (2.16a) provides the pressure field of the liquid flow and (2.16b) is used to derive the Rayleigh-Plesset-like equation for confined bubbles.

Assuming that the bubble gas undergoes an isothermal or adiabatic process, its pressure p_B is given as

$$p_B = p_{sat} + p_{g0}\left(\frac{R_{b0}}{R_b}\right)^{3\gamma}, \tag{2.17}$$

where p_{g0} is the initial partial pressure of the bubble gas and γ is the polytropic index of the bubble gas. Note that $\gamma = 1$ for an isothermal process and $\gamma > 1$ for an adiabatic process. For an isothermal process, temperature and chemical composition within the bubble are assumed to be uniform, with water vapor freely entering the bubble with minimal change in surface temperature. For an adiabatic process, the bubble motion is much faster so that the mass and thermal transfer may be neglected.

The liquid pressure at the bubble surface p_{lb} is given by the dynamics boundary condition on the bubble surface

$$p_{lb} = -p_A + p_B - \frac{2\sigma}{R_b} - 4\mu\frac{\dot{R}_b}{R_b}, \tag{2.18}$$

where σ is surface tension at the bubble surface and p_A is the pressure of the acoustic wave at the position of the bubble, when it is subject to an acoustic wave.

Substituting (2.4), (2.11), and (2.18) into (2.16b) yields the Rayleigh-Plesset-like equation for a confined bubble as follows:

$$\begin{aligned}\rho\left(R_b\ddot{R}_b + 2\dot{R}_b^2\right)\left(1 - \frac{R_b}{R_c}\right) - \rho\frac{\dot{R}_b^2}{2}\left(1 - \left(\frac{R_b}{R_c}\right)^4\right) &= -p_{l0} - p_A + p_{sat} + p_{g0}\left(\frac{R_{b0}}{R_b}\right)^{3\gamma} \\ &\quad - \frac{2\sigma}{R_b} - 4\mu\frac{\dot{R}_b}{R_b} + K\frac{R_b^3 - R_{b0}^3}{R_{c0}^3}.\end{aligned} \tag{2.19}$$

Equations (2.12) and (2.19) are the governing equations for $R_b(t)$ and $R_c(t)$. As $R_{c0} \to \infty$, the above equation reduces to the Rayleigh-Plesset equation for a bubble in an unbounded liquid as follows:

$$\begin{aligned}\rho R_b\ddot{R}_b + \rho\frac{3\dot{R}_b^2}{2} &= -p_{l0} - p_A + p_{sat} + p_{g0}\left(\frac{R_{b0}}{R_b}\right)^{3\gamma} \\ &\quad - \frac{2\sigma}{R_b} - 4\mu\frac{\dot{R}_b}{R_b}.\end{aligned} \tag{2.20}$$

III. NATURAL FREQUENCY OF A BUBBLE IN A CONFINEMENT

In the linear analysis for obtaining the natural frequency, we set $p_A = 0$ and

$$R_b(t) = R_{b0}\left(1 + \varepsilon R_{b1}e^{i\omega t}\right) \text{ with } \varepsilon << 1,\ R_{b1} = O(1). \tag{3.1}$$

Note here that R_{b0} and R_{c0} should be the equilibrium radii of the bubble and cavity, respectively, when the oscillation of a bubble at small amplitude is considered for calculating the natural frequency.

We substitute (3.1) into (2.19), expand all terms in the equation obtained in the Taylor series in terms of ε, and equal the first two order terms in ϵ on both sides, respectively, yielding

$$p_{sat} + p_{g0} = p_{l0} + \frac{2\sigma}{R_{b0}}, \tag{3.2a}$$

$$\rho\omega^2 R_{b0}^2(1 - \alpha) - 4\mu i\omega - \left(3\gamma p_{g0} + 3K\alpha^3 - \frac{2\sigma}{R_{b0}}\right) = 0, \tag{3.2b}$$

where $\alpha = R_{b0}/R_{c0}$ is the confinement ratio. Equation (3.2a) is the equilibrium condition, which is the same as that for a bubble in an unbounded liquid. The solutions for ω to (3.2b) are

$$\omega_{1,2} = \frac{2i\mu \pm \sqrt{-4\mu^2 + \rho R_{b0}^2 (1-\alpha)\left(3\gamma p_{g0} + 3K\alpha^3 - 2\sigma/R_{b0}\right)}}{\rho R_{b0}^2 (1-\alpha)}. \tag{3.3}$$

If the argument of the square root in (3.3) is positive, the amplitude of oscillation decays exponentially in the form of $\exp(-\beta t)$, where $\beta = 2\upsilon R_{b0}^{-2}(1 - R_{b0}/R_{c0})^{-1} > 0$, with υ being the kinematic viscosity of the liquid. The bubble is thus stable when it is subject to small disturbances, undergoing damping oscillation with the decay rate increasing due to the effects of confinement.

The natural angular frequency ω_n of oscillation of a bubble in a cavity is thus given as

$$\omega_n = \frac{\sqrt{-4\mu^2 + \rho R_{b0}^2 (1-\alpha)\left(3\gamma (p_{l0} - p_{sat}) + 3K\alpha^3 + 2(3\gamma - 1)\sigma/R_{b0}\right)}}{\rho R_{b0}^2 (1-\alpha)}. \tag{3.4}$$

A few features can be drawn from the formula: the natural frequency increases with the confinement ratio $\alpha = R_{b0}/R_{c0}$ and the effective bulk modulus K of the confinement. Like in an unconfined bubble, the natural frequency decreases with the equilibrium radius R_{b0} of the bubble and viscosity, but increases with the surface tension σ, the polytropic index γ of the bubble gas, and the difference $p_{l0} - p_{sat}$ of the ambient pressure and the saturated vapor pressure. As R_{c0} approaches infinity, (3.4) reduces to the natural frequency of a bubble in an infinite fluid as follows:[20]

$$\omega_{n\infty} = \frac{1}{R_{b0}^2}\sqrt{-4\upsilon^2 + \frac{R_{b0}^2}{\rho}\left(3\gamma (p_{l0} - p_{sat}) + 2(3\gamma - 1)\frac{\sigma}{R_{b0}}\right)}. \tag{3.5}$$

The dimensionless form of (3.4) is given as follows:

$$\omega_{n*} = \frac{1}{1-\alpha}\sqrt{-4Re^{-2} + (1-\alpha)\left(3\gamma \operatorname{sgn}(\Delta p) + 3K_*\alpha^3 + 2(3\gamma - 1)\sigma_*\right)}, \tag{3.6}$$

where $K_* = K/|\Delta p|$, $\sigma_* = \sigma/(R_{b0}|\Delta p|)$, $Re = R_{b0}\sqrt{|\Delta p|/\rho}/\upsilon$, with υ being the kinematic viscosity of the liquid, and sgn(Δp) is equal to 1, −1 as $\Delta p > 0$, < 0, respectively. The dimensionless frequency increases with the confinement ratio α, Reynolds number Re, the dimensionless effective bulk modulus K_* of the confinement, and the dimensionless surface tension σ_*.

If the effects of surface tension and viscosity are negligible, i.e., $Re >> 1$ and $\sigma_* << 1$, (3.6) becomes

$$\omega_n = \frac{1}{R_{b0}}\sqrt{\frac{3\gamma (p_{l0} - p_{sat}) + 3K\alpha^3}{\rho (1-\alpha)}}. \tag{3.7}$$

Under this situation, the dependency of both $\omega_n R_{b0}$ and $\omega_n R_{c0}$ in terms of R_{b0} and R_{c0} is through their ratio α, as observed in the experiments,[30]

$$\omega_n R_{b0} = \sqrt{\frac{3\gamma (p_{l0} - p_{sat}) + 3K\alpha^3}{\rho (1-\alpha)}},$$
$$\omega_n R_{c0} = \frac{1}{\alpha}\sqrt{\frac{3\gamma (p_{l0} - p_{sat}) + 3K\alpha^3}{\rho (1-\alpha)}}. \tag{3.8}$$

The effective bulk modulus of the confinement may be as high as $K = O(1)$ GPa.[30] If $K\alpha^3 >> \gamma |p_0 - p_{sat}|$, we have

$$\omega_n = \frac{\sqrt{3K/\rho}}{R_{b0}}\alpha^{3/2}(1-\alpha)^{-1/2}. \tag{3.9}$$

Under this situation, the oscillation frequency of a bubble depends on the liquid density ρ, the bubble size R_{b0}, the effective bulk modulus K, and the confinement ratio α. The natural frequency is in the MHz range as $K = O$(GPa), $R_{b0} \leq O$(mm), and $\alpha = O(1)$.

IV. COMPARISON WITH EXPERIMENTS

Vincent *et al.*[30–32] performed a series of experiments for the dynamics of a cavitation bubble in water in a microcavity confined by a stiff pHEMA hydrogel. The cavitation bubble was generated by a negative pressure of high magnitude of about $p_0 - p_{sat} = -20$ MPa. The radius of the cavity was in the range of R_{c0} = 15-200 μm and the ratio of the bubble equilibrium radius to the initial cavity radius, $\alpha = R_{b0}/R_{c0}$, was fixed at a value. Vincent *et al.*[30] set the value of α to 0.28 (solid line) from the experimental data, and it is reset to 0.265 (dashed line) since the latter fits better with the experimental data, as shown in Fig. 2(a). The bulk modulus of the confinement was estimated as K_c = 1 GPa.[30] The other parameters used are ρ = 988 kg·m^{-3}, μ = 0.001 Pa s, and σ = 0.07 N m^{-1}. It is easy to verify using (2.19) and (3.4) that the viscous and surface tension effects are negligible for the experimental cases.

To validate our model, we performed the calculations for the experimental cases using the present theory and compared to the experimental results for both the natural frequency and the time history of the oscillating bubble radius. As shown in Figs. 2(b) and 2(c), the theoretical results agree very well with the experiments. The agreement for the natural frequency is for all the ranges of R_{c0} of the experiments, as shown in Fig. 2(b). The agreement for the bubble radius history is for both the first-cycle and second-cycle of oscillation [Fig. 2(c)]. The initial conditions used in calculating the bubble radius history shown in Fig. 2(c) are R_{b0} = 1 μm and $\dot{R}_b(0) = 0$, where the over dot denotes the derivative in time t.

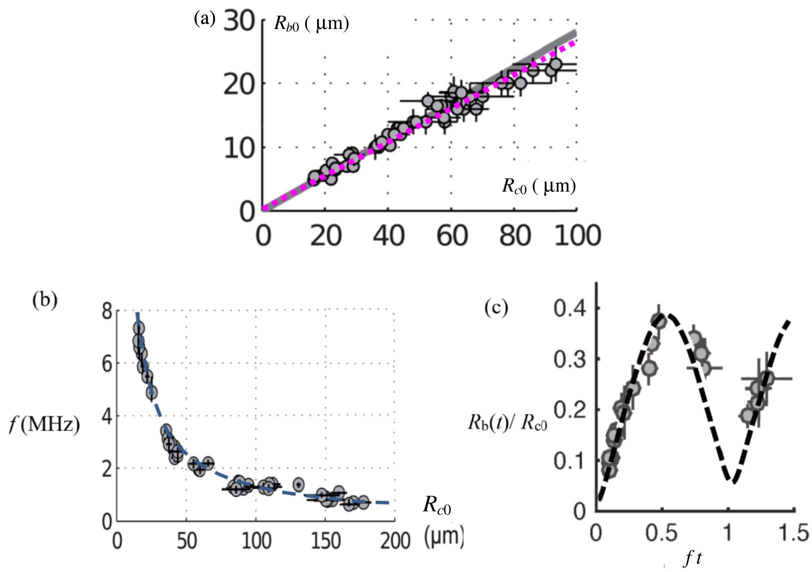

FIG. 2. Comparison of the present theory (dashed line) and experiments (gray circles)[30] for bubble dynamics in a microcavity confined by a stiff pHEMA hydrogel for R_{c0} = 15-200 μm, R_{b0}/R_{c0} = 0.265, ρ = 988 kg m^{-3}, μ = 0.001 Pa·s, σ = 0.07 N m^{-1}, and K = 1 GPa. (a) The equilibrium bubble radius R_{b0} versus the initial cavity radius R_{c0} recorded in the experiments. Their ratio $\alpha = R_{b0}/R_{c0}$ was set to 0.28 (solid line) by Vincent *et al.*[30] and was reset to 0.265 (dashed line). (b) The natural frequency f of oscillation of the bubble as a function of the initial microcavity radius R_{c0} and (c) the transient radius of the bubble, scaled by R_{c0}, as a function of the scaled time after cavitation. The initial conditions used in the calculation in (c) are R_{b0} = 1 μm, $\dot{R}_b(0)$ = 0, and $p_{l0} - p_{sat}$ = −20 MPa.

The discrepancies between the experiments and theory should be comparable with the measurement errors in the experiments.[30] The comparisons suggest that the theory predicts reasonable results for both the natural frequency of oscillation and the time history of the bubble radius for the nonlinear transient oscillation of a fully confined bubble.

V. NATURAL FREQUENCY

This section is concerned with the natural frequency of oscillation of a spherical bubble in a spherical cavity. Figure 3(a) shows the ratio $\omega_n/\omega_{n\infty}$ of the natural frequency for a bubble in a confinement and that in an infinite fluid, as a function of the confinement ratio R_{c0}/R_{b0} for the effective bulk modulus $K = 10^4$, 10^7, and 10^8 Pa, respectively. The remaining parameters are chosen as R_{b0} = 2.5 μm, p_{sat} = −2.3 kPa, p_{l0} = 100 kPa, ρ = 1000 kg m^{-3}, μ = 0.001 Pa s, and σ = 0.05 N m^{-1}. The natural frequency of oscillation for a bubble in an elastic confinement is larger, by an order of magnitude, than that in an unbounded liquid. This is due to the fact that the bubble oscillation is constrained by the confinement and the compressibility of the liquid is low. The effects of the confinement decrease rapidly with the cavity size but are still significant for a large cavity at R_{c0}/R_{b0} = 10, since a small deformation of the cavity is associated with appreciable pressure change for the liquid in the confinement.

Figure 3(b) shows the natural frequency versus the effective bulk modulus K of the confinement for R_{c0}/R_{b0} = 2.5, 3, and 4, respectively. The natural frequency increases substantially with the effective bulk modulus K of the elastic confinement, since the pressure of the liquid at the confinement is proportional to K, which reflects the stiffness of the oscillation system.

VI. TRANSIENT BUBBLE DYNAMICS IN A CONFINEMENT

Transient oscillation of a confined bubble starting from a non-equilibrium state is investigated in this section. The case considered is for a bubble at its maximum radius initially with the initial conditions R_{b0} = 5 μm, $\dot{R}_b(0)$ = 0, and p_{g0*} = 0.001. The remaining parameters are chosen as p_{l0} = 100 kPa, p_{sat} = 2300 Pa, ρ = 1000 kg m^{-3}, μ = 0.001 Pa s, and σ = 0.05 N m^{-1}.

Figure 4(a) compares the time histories of the radius of a bubble in a cavity at different sizes for R_{c0}/R_{b0} = 4, 7, and ∞, respectively, where $R_{c0}/R_{b0} = \infty$ is for an unbounded domain. Starting with collapse initially, it undergoes a damped oscillation due to the viscous damping effects. The amplitude of oscillation decreases with time, when the maximum radius decreases and the minimum radius increases. The frequency does not change significantly with time. The radius reaches an equilibrium value ultimately. The amplitude and period of oscillation decrease significantly due to the presence of the elastic confinement, reaching a larger equilibrium radius in a much longer period. These effects of confinement increase

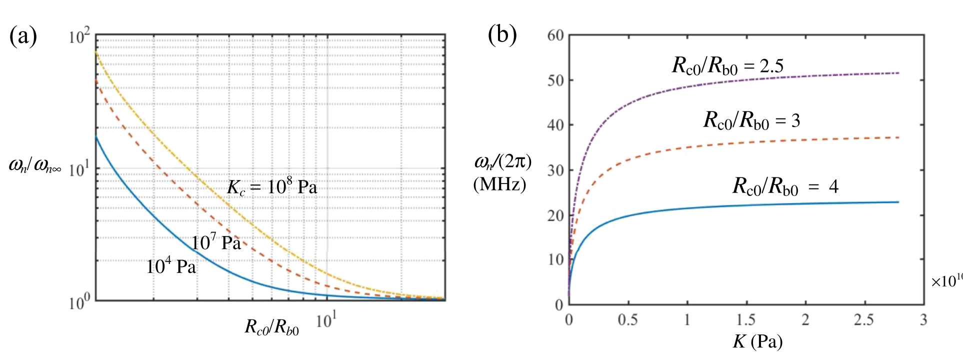

FIG. 3. (a) The ratio $\omega_n/\omega_{n\infty}$ of the natural frequency of oscillation for a bubble in a confinement to that in an infinite fluid as a function of R_{c0}/R_{b0} for $K = 10^4$, 10^7, and 10^8 Pa, respectively, and (b) the natural frequency as a function of K for R_{c0}/R_{b0} = 2.5, 3, and 4, respectively. The remaining parameters used are R_{b0} = 2.5 μm, p_{sat} = −2.3 kPa, p_{l0} = 100 kPa, ρ = 1000 kg m^{-3}, μ = 0.001 Pa s, and σ = 0.05 N m^{-1}.

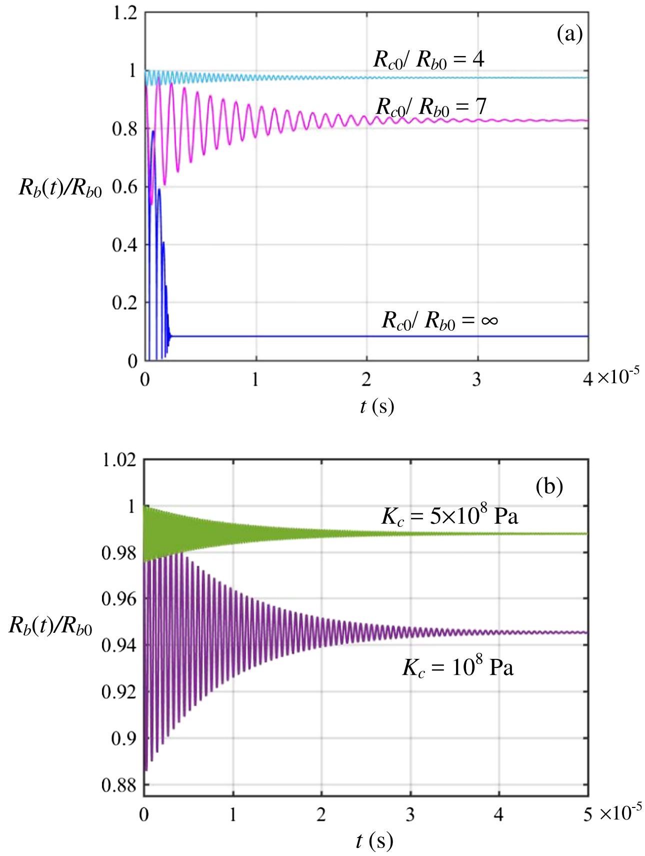

FIG. 4. Radius history for a transient bubble oscillating in a cavity for (a) R_{c0}/R_{b0} = 4, 7, and ∞ and $K = 10^8$ Pa, and (b) R_{c0}/R_{b0} = 4 and $K = 10^8$, 5×10^8 Pa. The remaining parameters are $R_{b0} = 5\ \mu$m, $p_{l0} = 10^5$ Pa, $p_{sat} = 2300$ Pa, $p_{g0*} = 0.001$, $\rho = 1000$ kg m^{-3}, $\mu = 0.001$ Pa s, and $\sigma = 0.05$ N m^{-1}.

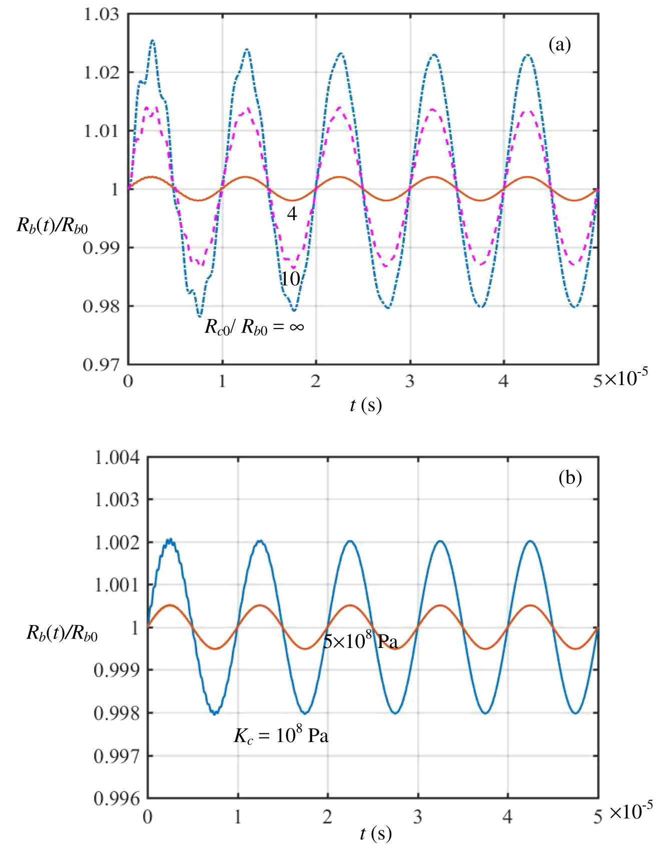

FIG. 5. Radius history for a bubble in a confinement subject to an acoustic wave for (a) R_{c0}/R_{b0} = 4, 10, ∞ and $K_c = 10^8$ Pa, and (b) R_{c0}/R_{b0} = 4 and $K = 10^8$, 5×10^8 Pa. The remaining parameters are $R_{b0} = 5\ \mu$m, $\dot{R}_b(0) = 0$, $p_{l0} = 10^5$ Pa, $p_{sat} = 2300$ Pa, $p_{g0} = 117.7$ kPa, $\rho = 1000$ kg m^{-3}, $\mu = 0.001$ Pa s, $\sigma = 0.05$ N m^{-1}, $p_a = 10^4$ Pa, and $\omega_a = 100$ kHz.

for a smaller cavity. This is because that a part of the energy of the bubble system is transmitted into the surrounding solid through the work done by the liquid pressure at the interface between the liquid and solid.

Figure 4(b) displays the effects of the effective bulk elastic modulus of the confinement for $R_{c0}/R_{b0} = 4$ and $K = 10^8$, 5×10^8 Pa. The amplitude and period of oscillation decrease with the effective bulk elastic modulus K of the confinement, while the equilibrium radius increases with K.

VII. OSCILLATION OF A CONFINED BUBBLE SUBJECT TO AN ACOUSTIC WAVE

We next consider a bubble, having the initial radius $R_{b0} = 5\ \mu$m, oscillating in a cavity subject to an acoustic wave. The acoustic pressure at the location of the bubble is characterized by amplitude p_{ac} and driving frequency ω_d,

$$p_A(t) = p_{ac}\sin(\omega_d t). \tag{7.1}$$

We choose p_{ac} = 10 kPa and ω_d = 100 kHz. Here, we assume that the wavelength λ of the acoustic wave is large compared to the bubble radius. In fact, for the case considered $\lambda = 2\pi c/\omega_d \approx 94$ mm $>> R_{b0}$. The other parameters are chosen to be the same as in Fig. 4 except that the initial gas pressure of the bubble is set at the equilibrium value p_{g0} = 117.7 kPa, so that the bubble is at the equilibrium state before the acoustic wave is activated.

Figure 5(a) compares the results of a bubble in cavities with different sizes for R_{c0}/R_{b0} = 4, 10, and ∞, respectively. The bubble did not oscillate harmonically for first few cycles. This is due to the transient features associated with the initial condition $\dot{R}_b(0) = 0$ set for the calculations, but $\dot{R}_b(0) \neq 0$ at the equilibrium radius for a steady oscillation. After the first few cycles, the bubble oscillates in a stable way around its equilibrium radius at the frequency of the driving wave for both confined and unconfined cases, when the energy absorbed by the bubble system from the acoustic wave is balanced by the energy damped due to the viscous effects and transmitted to the solid through the deformation of the confinement. The presence of confinement decreases the amplitude of oscillation significantly and decreases more for a smaller cavity. Figure 5(b) shows that the oscillation amplitude decreases significantly with the effective bulk elastic modulus of the confinement.

VIII. SUMMARY AND CONCLUSIONS

In this article, the Rayleigh-Plesset-like equation is derived for a bubble in a liquid in a cavity fully entrapped in an elastic solid. It is assumed that the variations of the liquid volume and the cavity volume are proportional to the variation of the liquid pressure at the cavity surface. Using the linear stability analysis, we show that the bubble undergoes stable damping oscillation subject to small disturbances and the damping rate accelerates due to the presence of the confinement. The natural frequency of oscillation is obtained analytically. The theory agrees well with the recent experimental data in terms of the natural frequency of oscillation and the time history of the radius of a transient bubble starting from a non-equilibrium state.

The natural frequency of oscillation, transient oscillation, and stable oscillation is analyzed in terms of the confinement size and elasticity of the surrounding solid. The following features/phenomena are noticed.

(i) The natural frequency of oscillation for a bubble in an elastic confinement is larger, in order of magnitude, than that in an unbounded liquid.
(ii) The amplitude and period of oscillation of a transient bubble decrease significantly due to the presence of the elastic confinement, reaching a steady state in a much longer period and at a larger equilibrium radius.
(iii) When subject to an acoustic wave, a bubble in a confinement oscillates with the frequency of the wave like that in an unbounded liquid, but the amplitude of oscillation decreases significantly due to the confinement.
(iv) The effects described above increase with the bulk modulus of the confinement and decrease rapidly with the cavity size but are still significant for a large cavity whose size is an order of magnitude larger than the bubble, since a small deformation of the cavity is associated with appreciable pressure change for the liquid in a confined cavity.

To facilitate the mathematical analysis, the bubble and the cavity are assumed to be spherical and concentric. The obtained features for confined bubbles for the special case are expected to be qualitatively correct for non-spherical bubbles/cavities and non-concentric cases. The results obtained can be used for validating the numerical models for more general situations.

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[1] L. Rayleigh, "On the pressure developed in a liquid during the collapse of a spherical cavity," Philos. Mag. **34**, 94–98 (1917).
[2] M. S. Plesset and A. Prosperetti, "Bubble dynamics and cavitation," Annu. Rev. Fluid Mech. **9**, 145–185 (1977).
[3] Z. C. Feng and L. G. Leal, "Nonlinear bubble dynamics," Annu. Rev. Fluid Mech. **29**, 201–243 (1997).
[4] Q. X. Wang and J. R. Blake, "Non-spherical bubble dynamics in a compressible liquid. Part 1. Travelling acoustic wave," J. Fluid Mech. **659**, 191–224 (2010).
[5] Q. X. Wang and J. R. Blake, "Non-spherical bubble dynamics in a compressible liquid. Part 2. Acoustic standing wave," J. Fluid Mech. **679**, 559–581 (2011).
[6] Q. X. Wang, "Local energy of a bubble system and its loss due to acoustic radiation," J. Fluid Mech. **797**, 201–230 (2016).
[7] S. G. Zhang, J. H. Duncan, and G. L. Chahine, "The final stage of the collapse of a cavitation bubble near a rigid wall," J. Fluid Mech. **257**, 147–181 (1993).
[8] Q. X. Wang and K. Manmi, "Microbubble dynamics near a wall subjected to a travelling acoustic wave," Phys. Fluids **26**, 032104 (2014).
[9] Q. X. Wang, K. Manmi, and K. K. Liu, "Cell mechanics in biomedical cavitation," Interface Focus **5**(5), 20150018 (2015).
[10] S. Martynov, E. Stride, and N. Saffari, "The natural frequencies of microbubble oscillation in elastic vessels," J. Acoust. Soc. Am. **126**(6), 2963–2972 (2009).
[11] Q. X. Wang, "Numerical modelling of violent bubble motion," Phys. Fluids **16**(5), 1610–1619 (2004).
[12] Q. X. Wang, "Non-spherical bubble dynamics of underwater explosions in a compressible fluid," Phys. Fluids **25**, 072104 (2013).
[13] Y. L. Liu, P. Wang, Q. X. Wang, and A. M. Zhang, "The motion of a 3D toroidal bubble and its interaction with a free surface near an inclined wall," Phys. Fluids **28**(12), 122101 (2016).
[14] J. H. Duncan, C. D. Milligan, and S. G. Zhang, "On the interaction between a bubble and a submerged compliant structure," J. Sound Vib. **197**(1), 17–44 (1996).
[15] R. Han, S. Li, A. M. Zhang, and Q. X. Wang, "Numerical modelling for three dimensional coalescence of two bubbles," Phys. Fluids **28**, 062104 (2016).
[16] F. Hamaguchi and K. Ando, "Linear oscillation of gas bubbles in a viscoelastic material under ultrasound irradiation," Phys. Fluids **27**, 113103 (2015).
[17] S. J. Lind and T. N. Phillips, "Spherical bubble collapse in viscoelastic fluids," J. Non-Newtonian Fluid Mech. **165**, 56–64 (2010).
[18] J. R. Blake and D. C. Gibson, "Cavitation bubbles near boundaries," Annu. Rev. Fluid Mech. **19**, 99–123 (1987).
[19] C. E. Brennen, *Cavitation and Bubble Dynamics* (Oxford University Press, 1995).
[20] T. Leighton, *The Acoustic Bubble* (Academic Press, London, 1994).
[21] W. Lauterborn and T. Kurz, "Physics of bubble oscillations," Rep. Prog. Phys. **73**, 106501 (2010).
[22] J. Rodríguez-Rodríguez, A. Sevilla, C. Martínez-Bazán, and J. M. Gordillo, "Generation of microbubbles with applications to industry and medicine," Annu. Rev. Fluid Mech. **47**, 405–429 (2015).
[23] A. Prosperetti, "Vapor bubbles," Annu. Rev. Fluid Mech. **49**, 221–248 (2017).
[24] M. T. Tyree and J. S. Sperry, "Vulnerability of xylem to cavitation and embolism," Annu. Rev. Plant Physiol. Plant Mol. Biol. **40**, 19–36 (1989).
[25] B. Choat *et al.*, "Global convergence in the vulnerability of forests to drought," Nature **491**, 752–755 (2012).
[26] H. J. Schenk, K. Steppe, and S. Jansen, "Nanobubbles: A new paradigm for air-seeding in xylem," Trends Plant Sci. **20**, 199–205 (2015).
[27] X. Noblin, N. O. Rojas, J. Westbrook, C. Llorens, M. Argentina, and J. Dumais, "The fern sporangium: A unique catapult," Science **335**, 1322 (2012).
[28] E. Roedder and R. J. Bodnar, "Geologic pressure determinations from fluid inclusion studies," Annu. Rev. Earth Planet. Sci. **8**, 263–301 (1980).
[29] D. Marti, Y. Krüger, D. Fleitmann, M. Frenz, and J. Rička, "The effect of surface tension on liquid-gas equilibria in isochoric systems and its application to fluid inclusions," Fluid Phase Equilib. **314**(0), 13–21 (2012).
[30] O. Vincent, P. Marmottant, S. R. Gonzalez-Avila, K. Ando, and C.-D. Ohl, "The fast dynamics of cavitation bubbles within water confined in elastic solids," Soft Matter **10**, 1455–1461 (2014a).
[31] O. Vincent, P. Marmottant, P. A. Quinto-Su, and C.-D. Ohl, "Birth and growth of cavitation bubbles within water under tension confined in a simple synthetic tree," Phys. Rev. Lett. **108**(18), 184502 (2012).
[32] O. Vincent, D. A. Sessoms, E. J. Huber, J. Guioth, and A. D. Stroock, "Drying by cavitation and poroelastic relaxations in porous media with macroscopic pores connected by nanoscale throats," Phys. Rev. Lett. **113**(13), 134501 (2014b).
[33] G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, 1967), pp. 137 and 142.